Astrophysics Assignment; Kramers' Opacity Law

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1 Transport of Energy

The energy the star radiates from its surface originates from its central region. There are different ways in which a star transports this energy. Mostly it uses radiation, conduction and convection. In each of the methods either particles or larger amounts of material are exchanged between hotter and cooler parts and their mean free path together with the temperature gradient of the surroundings will play a decisive role. The method of energy transport that will interest us the most is via radiation.

1.1 Radiative Transport of Energy

1.1.1 Basic Estimates

Let us first estimate the mean free path l_{ph} of a photon at an average point inside a star like the sun:

$$l_{\mathrm{ph}} = \frac{1}{\kappa\rho}, \tag{1}$$

where κ is a *mean absorption coefficient* (a radiative cross-section per unit mass averaged over frequency). Stellar matter is very opaque ($\kappa \approx 1$ cm^2 g^{-1}), while typical values of electron scattering are of order $\kappa \approx 0.4$ cm^2 g^{-1}. Using this and the mean density of matter in the sun, $\bar{\rho}_\odot = 1.4$ g cm^{-3}, we obtain a mean free path of only

$$l_{\mathrm{ph}} \approx 2\,\mathrm{cm}.$$

1.1.2 Diffusion of Radiative Energy

The above estimates have shown that for radiative transport in stars the mean free path l_{ph} of the "transporting particles" (photons) is very small compared to the characteristic length R (stellar radius) over which the transport extends: $l_{ph}/R_\odot \approx 3 \times 10^{-11}$. In this case, the transport can be treated as a diffusion process, which yields an enormous simplification of the formalism.

the diffusive flux j of particles between places of different particle density n is given by

$$j = -D\nabla n, \tag{2}$$

where D is coefficient of diffusion,

$$D = \frac{1}{3} v l_{\mathrm{p}}, \tag{3}$$

determined by the average values of mean velocity v and mean free path l_{p} of the particles.

In order to obtain the corresponding diffusive flux of radiative energy F, we replace n by the energy density of radiation U,

$$U = aT^4, \tag{4}$$

v by the velocity of light c, and l_{p} by l_{ph} according to equation (1). $a = 7.57 \times 10^{-15}$ erg cm^{-3} K^{-4} is the *radiation-density constant*.

Owning to the spherical symetry of the problem, F has only a radial component $F_r = |F| = F$ and ∇U reduces to the derivate in the radial direction

$$\frac{\partial U}{\partial r} = 4aT^3 \frac{\partial T}{\partial r}, \tag{5}$$

Then equations (2)–(3) give immediately that

$$F = -\frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{\partial T}{\partial r}. \tag{6}$$

$$F = -k_{\mathrm{rad}}\nabla T, \tag{7}$$

where

$$k_{\mathrm{rad}} = \frac{4ac}{3}\frac{T^3}{\kappa\rho} \tag{8}$$

represents the coefficient of conduction for this radiative transport.

We solve equation (6) for the gradient of the temperature and replace F by the usual local luminosity $l = 4\pi r^2 F$; then

$$\frac{\partial T}{\partial r} = -\frac{3}{16\pi ac}\frac{\kappa\rho l}{r^2 T^3} \tag{9}$$

Of course, this simple equation becomes invalid when one approaches the surface of the star since the mean free path becomes comparable with the remaining distance to the surface. Hence, the whole diffusion approximation brakes down.

1.1.3 The Rosseland Mean for κ_ν

The above equations are independent of the frequency ν; F and l are quantities integrated over all frequencies, so that the quantity κ must represent a "proper mean" over ν. We shall now prescribe a method for this averaging.

In general the absorption coefficient depends on the frequency ν. Let us denote this by adding a subscript ν to all quantities that become frequency dependent: $\kappa_\nu, l_\nu, D_\nu, U_\nu$, etc.

For the diffusive energy flux F_ν of radiation in the interval $[\nu, \nu + d\nu]$ we write now

$$F_\nu = -D_\nu \nabla U_\nu \tag{10}$$

with

$$D_\nu = \frac{1}{3}cl_\nu = \frac{c}{3\kappa_\nu\rho} \tag{11}$$

while the energy density in the same interval is given by

$$U_\nu = \frac{4\pi}{c}B(\nu, T) = \frac{8\pi h}{c^3}\frac{\nu^3}{e^{h\nu/kT} - 1}. \tag{12}$$

$B(\nu, T)$ denotes here the Planck function for the *intensity* of black-body radiation. From (12) we have

$$\nabla U_\nu = \frac{4\pi}{c}\frac{\partial B}{\partial T}\nabla T, \tag{13}$$

which together with (11) is inserted into (10), the latter then being integrated over all frequencies to obtain the total flux F:

$$F = -\left[\frac{4\pi}{3\rho}\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B}{\partial T}d\nu\right]\nabla T. \tag{14}$$

We have regained (7), but with

$$k_{\text{rad}} = \frac{4\pi}{3\rho}\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B}{\partial T}d\nu. \tag{15}$$

Equating this expression for k_{rad} with that in the averaged form of (8), we have immediately the proper formula for averaging the absorption coefficient:

$$\frac{1}{\kappa} = \frac{\pi}{acT^3}\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B}{\partial T}d\nu. \tag{16}$$

This is the so-called *Rosseland mean* (after Sven Rosseland).

Since

$$\int_0^\infty \frac{\partial B}{\partial T}d\nu = \frac{acT^3}{\pi}, \tag{17}$$

the Rosseland mean is formally the harmonic mean of κ_ν with the weighting function $\frac{\partial B}{\partial T}$, and it can simply be calculated, once the function κ_ν is known from atomic physics.

2 Opacity

For describing opacity function $\kappa(\rho, T)$ is used. Certain approximations are used for this function, however they never hold for the whole star and are used only in simplifying approaches.

2.1 Electron scattering

If an electromagnetic wave passes an electron, the electric field makes the electron oscilate. This electron represents a classical dipole that radiates in other directions, i.e. the electron scatters part of the energy of the incoming waves. The weakening of the original radiation due to scattering is equivalent to that by absorption, and we can describe it with κ_ν,

$$\kappa_\nu = \frac{8\pi}{3}\frac{{r_e}^2}{\mu_e m_u} = 0.20(1+X) \tag{18}$$

where r_e is the classical electron radius and X the mass fraction of hydrogen. $\mu_e = \frac{2}{1+X}$. Since κ_ν does not depend on the frequency, we immediately obtain the Rosseland mean of electron scattering:

$$\kappa_{SC} = 0.20(1+X)\,\mathrm{cm}^2\mathrm{g}^{-1} \tag{19}$$

The "Thomson scattering" just described neglects the exchange of momentum between electron and radiation. If this becomes important, then κ_ν will be reduced compared to the value given in equation (18), though this effect plays a role only at temperatures sufficiently high for the scattered photons to be very energetic. In fact during the scattering process the electron must obtain such a large momentum that its velocity is comparable to c for equation (19) to become a bad approximation.

2.2 Absorption Due to Free-Free Transmitions

If during its thermal motion a free electron passes an ion, the two charged particles form a system which can absorb and emit radiation. This mechanism is only effective as long as electron and ion are sufficiently close. The mean thermal velocity of the electron is $v \sim T^{1/2}$; therefore, if in a mass element the numbers of electrons and ions are fixed, the number of systems temporarily able to absorb is porportional to $T^{-1/2}$.

The absorption properties of such a system have been derived classicaly by Kramers, who calculated that the absorption coefficient per system is proportional to $Z^2\nu^{-3}$, where Z is the charge number of the ion. We therefore expect the absorption coefficient κ_ν of a given mixture of fully ionized matter to be

$$\kappa_\nu \sim Z^2 \rho T^{-1/2} \nu^{-3} \tag{20}$$

Here the factor ρ appears because for given mass element the probability that two particles are accidentally close together is proportional to the density.

If we use the equation (20), considering that a factor ν^α contained in κ_ν gives a factor T^α in κ, we find

$$\kappa_{\text{ff}} \sim \rho T^{-7/2} \tag{21}$$

All opacities of the form (21) are called *Kramers' opacities* and give only a classical approximation. We also omitted the factor Z^2 which appears in (20). The weighed sum over the values of Z^2 is taken into the constant of proportionality in (21), which then depends on the chemical composition.

3 The Assignment

Show that, if the frequency and temperature dependance of the mean free path for a photon is given by

$$l_\nu \propto \nu^3 T^{1/2}$$

then the frequency averaged opacity satisfies Kramers' law

$$\kappa \propto \rho T^{-3.5}$$

4 Solving the Problem

Let's assume the diffusion approximation,

$$D_\nu = \frac{1}{3} c l_\nu = \frac{c}{3\kappa_\nu \rho}$$

and

$$\kappa_\nu \sim Z^2 \rho T^{-1/2} \nu^{-3}. \tag{22}$$

We include equation (22) in the integral for the Rosseland's mean

$$\frac{1}{\kappa} = \frac{\pi}{acT^3} \int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B}{\partial T} d\nu. \tag{23}$$

Hence, we get

$$\frac{1}{\kappa} = \frac{\pi}{acT^3} \frac{T^{1/2}}{Z^2 \rho} \int_0^\infty \nu^3 \frac{\partial B}{\partial T} d\nu, \tag{24}$$

$$\frac{1}{\kappa} = \frac{\pi}{acT^3} \frac{T^{1/2}}{Z^2 \rho} \frac{\partial}{\partial T} \int_0^\infty \nu^3 B d\nu, \tag{25}$$

where B is

$$B(\nu, T) = \frac{2h}{c^2} \frac{\nu^3}{e^{h\nu/kT} - 1} \tag{26}$$

$$\frac{1}{\kappa} = \frac{2h\pi}{ac^3T^3} \frac{T^{1/2}}{Z^2 \rho} \frac{\partial}{\partial T} \int_0^\infty \frac{\nu^6}{e^{h\nu/kT} - 1} d\nu, \tag{27}$$

When we introduce the new variables:

$$u = \frac{h\nu}{kT}$$

$$du \propto \frac{1}{T} d\nu$$

the equation has the form

$$\frac{1}{\kappa} = \frac{2h\pi}{ac^3T^3} \frac{T^{1/2}}{Z^2 \rho} \frac{\partial}{\partial T} \frac{k^7}{h^7} \int_0^\infty \frac{T^6 u^6}{e^u - 1} T du. \tag{28}$$

$$\frac{1}{\kappa} \approx \frac{14h\pi}{ac^3T^3} \frac{T^{1/2}}{Z^2 \rho} \frac{k^7}{h^7} T^6 = \frac{14h\pi}{ac^3 Z^2 \rho} \frac{k^7}{h^7} T^{7/2}$$

As we can see, the result is

$$\kappa \propto \rho T^{-7/2}$$

Formally, the integral in the equation (28) can be expressed with Riemann's Zeta function:

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{u^{x-1}}{e^u - 1} du$$

where $x = 7$ in our case. Therefore, the value of the integral has the value of

$$\zeta(7) = 1.0083492774...$$

Since

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}.$$

References

[1] R. Kippenhahn, A. Weighert: *Stellar Structure and Evolution*, Springer (1994)

[2] www.mathworld.wolfram.com